

Semi-classical wormholes are unstable

Roman V. Buniy* and Stephen D.H. Hsu†

Institute of Theoretical Science, University of Oregon, Eugene OR 94703-5203

We show that Lorentzian (traversable) wormholes with semi-classical spacetimes are unstable. Semi-classicality of the energy-momentum tensor of the exotic matter used to stabilize the wormhole implies localization of its wavefunction in phase space, leading to evolution according to the classical equations of motion. Previous results related to violation of the NEC then require that the matter is unstable to small perturbations.

I. INTRODUCTION

The construction of a Lorentzian wormhole [1] or time machine [2] requires violation of the null energy condition (NEC):

$$T_{\mu\nu}n^\mu n^\nu \geq 0, \quad (1)$$

where T is the matter energy-momentum tensor and n is any null-vector ($n_\mu n^\mu = 0$). For example, in a wormhole the throat geometry requires that converging null geodesics evolve into diverging null geodesics (imagine the trajectories followed by light rays traversing the wormhole). The Raychaudhuri equation for the expansion θ of a hypersurface orthogonal null congruence is [3]

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}n^\mu n^\nu, \quad (2)$$

where λ is an affine parameter, n^μ the tangent vector field, and $\sigma^{\mu\nu}$ the shear. The first two terms on the right-hand side of Eq. (2) are negative, and after using the Einstein equation to relate $R_{\mu\nu}$ and $T_{\mu\nu}$, we see that θ can increase only if the NEC is violated.

In this paper we consider the exotic matter necessary to stabilize a traversable wormhole throat (which may or may not be a time machine). Our analysis focuses on constraining its properties, using the fact that it violates the NEC and some additional assumptions about the throat spacetime.

It is important to note that both the magnitude of $T_{\mu\nu}$ and its violation of the NEC must be large in order to construct a useful wormhole. Roughly, a wormhole with throat diameter of order L requires energy-momentum densities of order $\sim l_P^{-2}L^{-2}$, where l_P is the Planck length. Taking L to be of order meters requires energy-momentum densities greater than fm^{-4} [1]. Similarly, it is easy to show using Eq. (2) that the time required to travel through the wormhole is determined by the degree of violation of the NEC. This travel time is $\gtrsim l_P^{-1}(-T_{\mu\nu}n^\mu n^\nu)^{-\frac{1}{2}}$, where $T_{\mu\nu}n^\mu n^\nu < 0$. For travel time of order 1 year, this requires $|T_{\mu\nu}n^\mu n^\nu| \sim \text{keV}^4$.

Thus, the exotic matter under consideration must exhibit large energy density and substantial violation of the NEC.

We define two types of wormholes (or time machines constructed using wormholes), henceforth referred to as “devices”: those with semi-classical spacetimes (type A) and those with strongly fluctuating spacetimes (type B). Clearly, type A devices are preferable to type B as they can by definition be controlled more precisely, and perhaps pose less risk to users in their operation. Unfortunately, we will show that all type A devices are unstable to small perturbations, so time travel or travel via wormhole is possible only via an intrinsically quantum device with a “fuzzy” spacetime. Essentially, type A devices require exotic matter which behaves classically (quantum effects are negligible in its dynamics), and classical matter which violates the NEC is unstable [4].

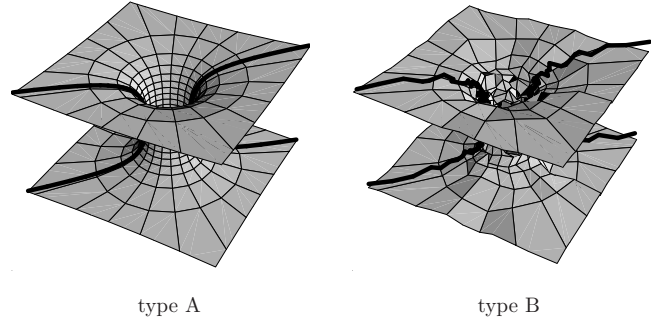


FIG. 1: Type A wormhole with semi-classical spacetime, and type B wormhole with large spacetime fluctuations induced by quantum matter.

By a semi-classical state $|a\rangle$ we mean one whose probability distribution is peaked about its central value in a particular basis, defined by some operator Z . In this case,

$$\langle a|Z^n|a\rangle = \langle a|Z|a\rangle^n \quad (3)$$

up to small corrections (assuming $\langle a|a\rangle = 1$), and we say that $|a\rangle$ is semi-classical with respect to the operator Z . This condition of semi-classicality is weaker than the usual one, which requires that Eq. (3) holds for all operators simultaneously.

A state might, according to our definition, be semi-classical with respect to Z but not with respect to some other operator Y (i.e., it might exhibit large fluctuations

*Electronic address: roman@uoregon.edu

†Electronic address: hsu@duende.uoregon.edu

in measurements of Y). For example, $|a\rangle$ might be an eigenstate of Z , while $[Y, Z] \neq 0$ (e.g., let Z be the momentum operator and Y the position operator). A central issue addressed in this paper is whether the time evolution of a device which gives rise to a semi-classical, non-zero $T_{\mu\nu}$ is well-approximated by the classical equations of motion, or equivalently, whether it is simultaneously semi-classical with respect to both the field operators ϕ and their conjugate momenta π .

If the time evolution of the exotic matter in the device is well-approximated by the classical equations of motion, we can apply a previous result [4], showing that in models built from gauge fields, scalars and fermions, any classical solution is unstable to small perturbations if it violates the NEC. Specifically, in Ref. [4] it was shown that the kinetic term of the effective Hamiltonian governing small fluctuations about a classical solution must exhibit a negative eigenvalue if the NEC is violated. Therefore, there exist infinitesimal perturbations involving local spatial gradients which lower the energy of the solution.

II. SPACETIME FLUCTUATIONS

In this section we discuss spacetime fluctuations [5], and argue that type A devices are necessarily semi-classical with respect to the matter energy-momentum tensor $T_{\mu\nu}$.

It is useful to distinguish between “active” and “passive” metric fluctuations. Active fluctuations are those involving quantum effects due to the graviton itself; these are negligible as long as the spacetime curvature is small compared to the Planck scale. We focus on passive fluctuations which are induced on the metric through the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ by fluctuations in $T_{\mu\nu}$. Clearly, in order for the spacetime to be semi-classical, the matter state must be semi-classical with respect to $T_{\mu\nu}$.

Because of ultraviolet divergences, the property of semi-classicality is scale-dependent. On sufficiently small length scales the passive fluctuations become arbitrarily large. Operator expectations as in Eq. (3) require regularization; for example, if $Z = T_{\mu\nu}$, it involves products of fields at the same point in spacetime. We assume a regulator such as point-splitting or smearing of the fields, controlled by a length scale σ . Short-distance contributions to operator averages are then of order σ^{-1} to the appropriate power, leading, e.g., to fluctuations

$$\Delta T_{\mu\nu} \equiv (\langle T_{\mu\nu} T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^2)^{\frac{1}{2}} \sim \sigma^{-4} \quad (4)$$

(no summation over indices). To investigate the effects of these fluctuations, we take σ to be some characteristic length scale associated with the spacetime or relevant experimental probes, such as test particles.

Minkowski spacetime is type A on length scales $\sigma > l_P$ (where l_P is the Planck length), even though $\langle 0|T_{\mu\nu}|0\rangle = 0$, so that $\Delta T_{\mu\nu} \sim \sigma^{-4} > \langle 0|T_{\mu\nu}|0\rangle$ for any σ . This is

because the effect of these fluctuations on the metric g is suppressed by l_P , as can be seen from the Einstein equations: $\Delta G_{\mu\nu} \sim l_P^2 \Delta T_{\mu\nu} \sim l_P^2 \sigma^{-4}$. Since $G_{\mu\nu}$ is constructed of objects with two derivatives of g , we see that deviations of g from flat space on length scales σ are of order $l_P^2 \sigma^{-2} \ll 1$. Borgman and Ford have studied the effect of vacuum energy fluctuations on null geodesics in Minkowski space, and found them to be undetectable [6].

Now consider a device whose spacetime is nontrivial — i.e., not flat space. If we take σ to be some characteristic length scale associated with the device spacetime, the regulated short-distance fluctuations are small compared to the central values of the energy momentum tensor: $\langle a|T_{\mu\nu}|a\rangle \gg \sigma^{-4}$. In fact, from the Einstein equations we expect $\langle a|T_{\mu\nu}|a\rangle \sim l_P^{-2} \sigma^{-2}$. In other words, the energy and momentum densities required to warp spacetime on length scale σ are much larger than σ^{-4} . (As mentioned, in the wormholes studied by Morris and Thorne in Ref. [1], for a throat size of order meters the required energy-momentum densities are of order fm^{-4} , provided perhaps by a cosmic string or similar object.) Then, the question of whether the spacetime is semi-classical depends not on the short-distance singularities in the operator products, but rather on whether the wavefunction of $|a\rangle$ is highly peaked about its central value on length scales of σ or larger. *The classicality of the spacetime is determined by the classicality of the energy-momentum tensor in the matter state $|a\rangle$.*

As an example, consider the Casimir vacuum $|c\rangle$, which describes the interior of a cavity of size L . The NEC is violated in the Casimir state, which has negative (renormalized) energy density: $\langle c|T_{00}|c\rangle \sim -1/L^4$. However, since the fluctuations ΔT are of order $1/\sigma^4$, they are larger than the central value even if we choose the largest possible length scale $\sigma \sim L$; the Casimir state never leads to a semi-classical energy-momentum tensor. Further, the Casimir state is problematic for construction of an interesting device, since the curvatures induced are only of order $l_P^2 L^{-4} \ll L^{-2}$. Spacetime within the cavity deviates from flat space only very slightly, and in an intrinsically fuzzy (highly fluctuating) manner.

III. ENERGY-MOMENTUM TENSOR AND PHASE SPACE

In this section we show that semi-classicality with respect to $T_{\mu\nu}$ implies semi-classicality with respect to the fields ϕ and their conjugate momenta π . Hence the wavefunction is localized in phase space, and the quantum evolution is closely approximated by the classical evolution. The proof proceeds in two steps. First, we note that at least two components of $T_{\mu\nu}$ do not commute with each other. Then, we derive a relation between the uncertainties in two such non-commuting operators and arbitrary functions of them. In the case of interest, the original pair is two non-commuting components of $T_{\mu\nu}$ and the functions map these operators into ϕ and π . The rela-

tion shows that small uncertainties in $T_{\mu\nu}$ imply small uncertainties in the phase space variables.

First, using the translation operator, it is easy to see that $F_{\mu\nu\rho\sigma}(x, x') = [T_{\mu\nu}(t, x), T_{\rho\sigma}(t, x')]$ is related to $F_{\mu\nu\rho\sigma}(x - y, x' - y)$ by a unitary transformation. Therefore, if for some x , $F_{\mu\nu\rho\sigma}(x, x') = 0$ for all x' , then $F_{\mu\nu\rho\sigma}(x, x') = 0$ identically. However, if that were the case we would have, in particular, $[T_{\mu\nu}(t, x), T_{00}(t, x')] = 0$ for all x and x' . Integrating over x' , we find that $T_{\mu\nu}(t, x)$ does not depend on time, which is possible only if $\mathcal{L} = \text{const.}$ Thus, in any non-trivial model, for any x there always exists an x' for which $F_{\mu\nu 00}(x, x') \neq 0$. In fact, $F_{\mu\nu\rho\sigma}(x, x')$ has support only for x near x' , since it is proportional to the delta function $\delta(x - x')$ or its spatial derivatives; this follows from the canonical commutation relation $[\phi(t, x), \pi(t, x')] = i\delta(x - x')$. Thus, there are at least two non-commuting components of the energy-momentum tensor.

Next, suppose we have a pair of non-commuting operators A and B , and consider a new operator which is an arbitrary function of them, $F = \mathcal{F}(A, B)$. The only limitation imposed on the function \mathcal{F} is that it can be expanded in a power series of its arguments. (Note that if A and B commute we cannot necessarily express an arbitrary F as a local function of them. For example, let $A = x$, $B = x^2$ and $F = p$.) It should be clear that if both A and B are semi-classical, then F is semi-classical too. Indeed, let us expand the operator F around its classical value $\bar{F} = \mathcal{F}(\bar{A}, \bar{B})$:

$$\begin{aligned} F &= \bar{F} + \bar{\mathcal{F}}_{\bar{A}}(A - \bar{A}) + \bar{\mathcal{F}}_{\bar{B}}(B - \bar{B}) \\ &+ \frac{1}{2}\bar{\mathcal{F}}_{\bar{A}\bar{A}}(A - \bar{A})^2 + \frac{1}{2}\bar{\mathcal{F}}_{\bar{B}\bar{B}}(B - \bar{B})^2 \\ &+ \frac{1}{2}\bar{\mathcal{F}}_{\bar{A}\bar{B}}[A - \bar{A}, B - \bar{B}] + \mathcal{O}(\Delta^3), \end{aligned} \quad (5)$$

where Δ is of order ΔA or ΔB . The fluctuation $(\Delta F)^2 = \langle F^2 \rangle - \langle F \rangle^2$ then is

$$\begin{aligned} (\Delta F)^2 &= \bar{\mathcal{F}}_{\bar{A}}^2(\Delta A)^2 + \bar{\mathcal{F}}_{\bar{B}}^2(\Delta B)^2 \\ &+ \bar{\mathcal{F}}_{\bar{A}}\bar{\mathcal{F}}_{\bar{B}}(\langle AB \rangle + \langle BA \rangle - 2\bar{A}\bar{B}) + \mathcal{O}(\Delta^3). \end{aligned} \quad (6)$$

For semi-classical A and B , the third term on the right hand side of Eq. (6) is as small as the first two terms. Thus F is semi-classical if A and B are.

If there is another operator $G = \mathcal{G}(A, B)$, we also expand it around its classical value $\bar{G} = \mathcal{G}(\bar{A}, \bar{B})$ and find the commutator

$$[F, G] = (\bar{\mathcal{F}}_{\bar{A}}\bar{\mathcal{G}}_{\bar{B}} - \bar{\mathcal{F}}_{\bar{B}}\bar{\mathcal{G}}_{\bar{A}})[A, B] + \mathcal{O}(\Delta^3). \quad (7)$$

Alternatively, in the semi-classical limit the commutator $i[F, G]$ turns into the Poisson bracket $\hbar\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}$. Since the brackets $\{\bar{\mathcal{F}}, \bar{\mathcal{G}}\}$ and $\{\bar{A}, \bar{B}\}$ are related via the Jacobian $J = \partial(\bar{\mathcal{F}}, \bar{\mathcal{G}})/\partial(\bar{A}, \bar{B})$, we arrive at Eq. (7) again.

Fluctuations of two semi-classical operators A and B satisfy the relation $\Delta A \Delta B \sim \hbar\langle[A, B]\rangle$. Since F and G are semi-classical too, writing the similar relation for them and combining the result with Eq. (7), we arrive at

$$\Delta F \Delta G \sim J \Delta A \Delta B. \quad (8)$$

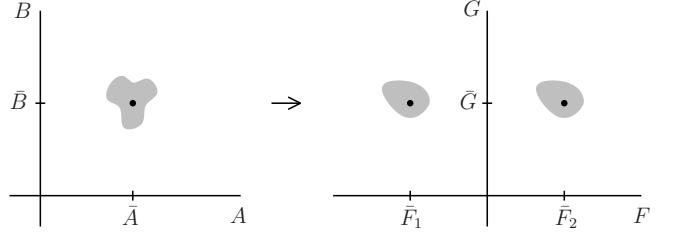


FIG. 2: The diagram illustrates the relation between localization in the (A, B) and (F, G) spaces, up to possible degeneracy.

To complete our proof, we identify the operators A, B with two non-commuting components of $T_{\mu\nu}$, and the operators F, G with ϕ, π . A state which has semi-classical energy-momentum tensor then must be localized in phase space, and therefore evolve according to the classical equations of motion. Note that if either of the commutators in Eq. (7) vanish, the Jacobian J would be either zero or divergent, and localization in (A, B) space would not imply localization in (F, G) space, or vice versa.

It is possible that degeneracies (either discrete or continuous) lead to multiple small regions in the (F, G) space which are mapped onto the same small region of the (A, B) space (see Fig. (2)). In other words, there may be inequivalent configurations in phase space with the same energy-momentum tensor. In such a case, the preceding equations should be modified by including the sums (or integrals, in the continuous case) over these regions. For example, if two states $|a_1\rangle$ and $|a_2\rangle$ with distinct values of $\langle\phi\rangle$ or $\langle\pi\rangle$ lead to the same $\langle T_{\mu\nu} \rangle$, we can obtain a semi-classical spacetime from a superposition of the two. Nevertheless, *each* component of the superposition is localized (behaves semi-classically) and will separately exhibit an instability.

IV. DISCUSSION

Classical systems which violate the NEC are unstable [4] in a particularly violent way: they can lower their energy by increasing local spatial field gradients. Therefore, the exotic matter used to stabilize a worm-hole throat must be quantum mechanical in nature; in other words, quantum effects must play an important role in its dynamics and time evolution. However, it is undesirable for a device to have a strongly fluctuating (non-semi-classical) spacetime. Such a device would presumably behave unpredictably and might transport its payload to an undesirable time or place. We have shown that semi-classicality of the spacetime is a strong enough condition to imply phase space localization of the wavefunction of the stabilizing matter. This means the time evolution of type A devices will be semi-classical and well-approximated by the classical equations of motion. Such devices are unstable in any region where the NEC is violated. Wormholes cannot be both predictable and stable.

Acknowledgments

This work was supported by the Department of Energy under DE-FG06-85ER40224.

-
- [1] M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988); J. L. Friedman, K. Schleich and D. M. Witt, Phys. Rev. Lett. **71**, 1486 (1993) [Erratum-ibid. **75**, 1872 (1995)]; D. Hochberg and M. Visser, Phys. Rev. D **56**, 4745 (1997); M. Visser and D. Hochberg, arXiv:gr-qc/9710001. For a review, see M. Visser, *Lorentzian wormholes: From Einstein to Hawking*, AIP, Woodbury USA, 1995.
- [2] S. W. Hawking, Phys. Rev. D **46**, 603 (1992).
- [3] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of space-time*, Cambridge University Press, 1973.
- [4] R. V. Buniy and S. D. H. Hsu, arXiv:hep-th/0502203.
- Similar instabilities in scalar models were studied by S. D. H. Hsu, A. Jenkins and M. B. Wise, Phys. Lett. B **597**, 270 (2004).
- [5] C. I. Kuo and L. H. Ford, Phys. Rev. D **47**, 4510 (1993); N. G. Phillips and B. L. Hu, Phys. Rev. D **55**, 6123 (1997); B. L. Hu and N. G. Phillips, Int. J. Theor. Phys. **39**, 1817 (2000); N. G. Phillips and B. L. Hu, Phys. Rev. D **62**, 084017 (2000); L. H. Ford and C. H. Wu, Int. J. Theor. Phys. **42**, 15 (2003).
- [6] J. Borgman and L. H. Ford, Phys. Rev. D **70**, 064032 (2004).